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# Covariant generalisation of Codazzi–Raychaudhuri and area change equations for relativistic branes

Elias Zafiris<sup>1</sup>

*Theoretical Physics Group, Imperial College, The Blackett Laboratory, London SW7 2BZ, UK*

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## Abstract

In this paper we derive the generalisation of Gauss–Codazzi, Raychaudhuri and area change equations for classical relativistic branes and multi-dimensional fluids in arbitrary background manifolds with metricity and torsion. The kinematical description we develop is fully covariant and based on the use of projection tensors tilted w.r.t. the brane worldsheets. ©1998 Elsevier Science B.V.

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## 1. Introduction

A large variety of physical systems is possible to be modelled as relativistic branes of an appropriate dimension propagating in a fixed background manifold. In general a  $(p - 1)$  brane is to be understood as a dynamical system defined in terms of fields with support confined to a  $p$ -dim worldsheet surface  $S$  in a background space–time manifold  $M$  of dimension  $n \geq p$  [1–3].

Stachel's idea of matter of multi-dimensional objects [4–6] is a generalisation of the notion of point particles to matter, whose elementary constituents are extensive objects. In turn a multi-dimensional fluid of extended objects on  $M$  is defined by a congruence of  $p$ -dim worldsheets of  $(p - 1)$ -dim branes.

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<sup>1</sup> Corresponding author. E-mail: e.zafiris@ic.ac.uk

A great deal of interest in brane models comes from cosmology, and especially from theories of structure formation in the early universe [7,8]. Essentially the relativistic theory of classical branes may be applied to vacuum defects produced by the Kibble mechanism [9], with interesting cosmological and astrophysical implications [10].

In recent years there has been a significant amount of work regarding the development of a kinematical description of deformations of the worldsheet spanned in the background manifold by a relativistic brane [11–16]. The main motivation for a proper kinematical description originates from a clear analogy. To be concrete, it is well established today that the proof of the existence of space–time singularities in general relativity relies on the consequences obtained from Raychaudhuri equations for geodesic congruences [17–19]. In brane theories the notion of the point particle and the associated with it worldline gets replaced by the notion of extensive objects with their corresponding worldsheets. Thus, in principle, it would be possible to derive the generalised Raychaudhuri equations for brane worldsheet congruences and arrive at analogous singularity theorems in classical relativistic brane theory.

The great majority of the earlier approaches have been expressed in a highly gauge dependent notation, as well as after the introduction of special reference systems, involving specifically adapted coordinates and frames, tailored to the embedding of particular surfaces, that require the simultaneous use of many different kinds of indices. On the other hand the advantages of a covariant kinematical description have been emphasised by Carter. Moving in this direction, Carter, based on traditional surface embedding theory [20–24], has developed a kinematical formalism which is, however, in his words, “designed to be a balance compromise between frame-dependent and tensorial expressions” [1,15,16,25–27]. Motivated by the significance of developing a proper mathematical machinery for the study of brane models, in this paper we construct a generalised fully covariant kinematical framework for branes and multi-dimensional fluids in manifolds with metricity and torsion, using projection tensor techniques.

Moreover, all the previous approaches impose major restrictions on the kind of brane models considered. More concretely, on the first place, the background manifold has to be endowed with a metric tensor which is at least invertible. On the second place, they assume that the background manifold connection has to be metric compatible and torsion free, thus excluding any discussion of brane models in the higher-dimensional manifolds of unified field theories, which use non-metric compatible and non-torsion free connections. Finally, they assume that the projection onto the brane worldsheet is normal, thus excluding the treatment of null brane worldsheets [28] where the normality condition is not satisfied. In this paper we remove these restrictive assumptions. One key idea is to define two different projection gradients which become equal for normal projections.

Using the generalised kinematical framework we construct, we manage to obtain the generalisations of the Gauss–Codazzi and Raychaudhuri equations, governing the behaviour of brane worldsheet congruences in manifolds with metricity and torsion, as well as the law governing their generalised area change. In this way we recover and greatly generalise the results of the current literature, for example those of Capovilla and Guven [13,14], and Carter [15,16], establishing an elegant covariant kinematical formalism at the same time.

The structure of the paper is as follows: In Section 2 we develop our covariant kinematical framework for brane worldsheet congruences. In more detail, in Section 2.1 we introduce the descriptive elements of a general brane model and present the projected index formalism as well as the geometrical objects notation we are going to use. In Section 2.2 we derive the kinematical quantities associated with the brane worldsheet projection tensor fields. In Section 2.3 we construct the intrinsic and extrinsic brane worldsheet projected covariant derivatives. In Section 2.4 we decompose the metricity and torsion tensors and derive the generalised Weingarden identity. In Section 2.5 we consider the decomposition of the Riemannian curvature w.r.t. the brane worldsheet and derive the generalisations of the Gauss–Codazzi and Raychaudhuri equations in covariant form. In Section 3 we study the brane worldsheet decomposition of the Lie derivatives of the various geometrical objects, and we obtain the law governing the generalised area change. Finally, we summarise and conclude in Section 4.

## 2. Covariant kinematical framework for brane worldsheet congruences

### 2.1. Preliminaries and notation

The basic descriptive element of a general  $(p - 1)$  brane model as localised on a  $p$ -dim worldsheet in an  $n$ -dim curved background manifold is the rank- $p$  operator  $Z^\mu_\nu$  of projection on the worldsheet. The projection tensor field  $Z$  assigns to each point  $P$  of the manifold a map  $Z(P) : T_P \rightarrow T_P$ , on the tangent space  $T_P$ , such that

$$Z(P)^2 = Z(P). \quad (1)$$

If  $I$  is the identity map, the tensor field

$$V = I - Z \quad (2)$$

is also a projection tensor field which we call the complement of  $Z$ . An immediate consequence of (1) and (2) is  $ZV = VZ = 0$ .

So long as there is a regular metric tensor we can expect the projection tensor field  $Z_{\mu\nu}$  to be normal or equivalently

$$Z_{\mu\nu} = Z_{\nu\mu}. \quad (3)$$

We note that the above condition cannot be imposed on the projection tensor onto a null brane worldsheet.

In cases where the normality condition (3) can be imposed, like the projection onto a timelike brane worldsheet, the rank- $p$  operator  $Z_{\mu\nu}$  of tangential projection onto the brane worldsheet represents the metric induced on the  $p$ -dim worldsheet by its embedding in the  $n$ -dim background manifold, whereas the complementary rank- $(n - p)$  operator  $V_{\mu\nu}$  of projection orthogonal to the worldsheet represents the projected metric on the  $(n - p)$ -dim quotient space if we consider a space–time filling congruence of brane worldsheets.

Moreover, the dimension of the projection tensor  $Z$  at the point  $P$  is defined to be the dimension of the subspace  $ZT_P$  which is the tangent space of the brane worldsheet. An immediate consequence of this definition is

$$p = Z^\alpha{}_\alpha, \quad n - p = V^\alpha{}_\alpha.$$

In turn, the projection tensor  $Z$  onto a null brane worldsheet is characterised by a projected metric tensor (or a projected inverse metric tensor) which is non-invertible and thus fails to define a metric on the subspace  $ZT_P$ .

Because  $Z$  and  $V$  have the same formal properties, it is possible to arrange all the expressions so that they are preserved by an exchange of  $Z$  and  $V$ . We have arranged the notation so that the exchange of a projection tensor by its complement can be carried out easily.

The projection of high-rank tensors related to the kinematical framework of branes often has lengthy expressions involving many index contractions. The expressions become much more elegant and transparent if we adopt a compact projected index notation formalism.

Thus we follow the conventions: Each tensor index that is meant to be contracted with an index on the projection tensor  $Z$  is marked by the symbol  $\wedge$ , while each index which is meant to be contracted with an index on the projection tensor  $V$  is marked by the symbol  $\vee$ . For example

$$X^{\hat{\gamma}\check{\alpha}}_{\hat{\delta}} := Z^\gamma{}_\rho V^\alpha{}_\sigma Z^\beta{}_\delta X^{\rho\sigma}{}_\beta. \quad (4)$$

We also notice the distinction between a tensor, which inhabits a particular projection subspace which we call entirely projected, and a tensor which is the result of projecting the corresponding background manifold tensor into that subspace. An entirely projected tensor carries a projection label for each of its tensor indices. The projection labels stand for projection tensors and indicate the projection identities which are associated with each tensor index. We can, by convention, abbreviate and use a single symbol ( $\alpha$ ) to stand for an index-label pair  $\hat{\alpha}$  or  $\check{\alpha}$ . We interpret the summation convention on a repeated and abbreviated symbol to imply a sum over both the visible index value  $\alpha$  and the invisible projection label, and denote it by  $[\alpha]$ . This notation has the advantage that a whole collection of entirely projected tensors is organised into a single geometrical object with appropriate labels according to the above conventions. Moreover, organising entirely projected tensors into geometrical objects has the advantage of making very compact expressions, which are identical in structure to familiar unprojected tensor expressions.

Let us agree that the symbol  ${}^P A^{(\rho)}_{(\mu)(\nu)}$  denotes the projected background manifold  $A^\rho{}_{\mu\nu}$  tensor and that the symbol  $A^{(\rho)}_{(\mu)(\nu)}$  denotes the entirely projected background manifold  $A^\rho{}_{\mu\nu}$  tensor. The projected  $A$  tensor geometrical object can be considered as a representation of the background manifold tensor  $A$ . In what follows it will become clear that whenever a background manifold tensor is defined in terms of covariant derivatives, its corresponding entirely projected geometrical object will be different from the geometrical object of the projected background manifold tensor.

In the following we use the definitions and notation of the projected and entirely projected geometrical objects to set a covariant kinematical framework for branes and

multi-dimensional fluids in manifolds having metricity and torsion, i.e. dealing with the most general case of brane models that can occur.

2.2. Kinematical quantities associated with the brane worldsheet projection tensor fields

The first step to set up a covariant kinematical framework for classical relativistic branes is to find a way to express the first derivative of the projection tensor field on the  $p$ -dim brane worldsheet as well as of its complement, in terms of tensors that obey projection identities.

We define the projection gradient to be the tensor:

$$Z^\alpha{}_\sigma Z^\rho{}_\beta \nabla_\rho Z^\sigma{}_\gamma = \nabla_{\hat{\beta}} Z^{\hat{\alpha}}{}_\gamma := z_\gamma{}^{\hat{\alpha}}{}_{\hat{\beta}}. \tag{5}$$

The definition of the projection gradient is projected explicitly on two of its three indices. However, we can easily show that it obeys the projection identity:

$$\nabla_{\hat{\beta}} Z^{\hat{\alpha}}{}_\gamma = \nabla_{\hat{\beta}} Z^{\hat{\alpha}}{}_{\check{\gamma}} \tag{6}$$

Another way to project the covariant derivative of a projection tensor yields the tensor:

$$Z^\sigma{}_\gamma Z^\rho{}_\beta \nabla_\rho Z^\alpha{}_\sigma = \nabla_{\hat{\beta}} Z^\alpha{}_{\check{\gamma}} := z^\alpha{}_{\check{\gamma}}{}_{\hat{\beta}}. \tag{7}$$

When a metric is available and  $Z$  is a normal projection tensor, this projection gradient is exactly the same as the previous one, with the indices appropriately raised and lowered.

In turn it obeys the projection identity

$$z^\alpha{}_{\check{\gamma}}{}_{\hat{\beta}} = z^{\check{\alpha}}{}_{\hat{\gamma}}{}_{\hat{\beta}}. \tag{8}$$

In addition to the projection gradient associated with the brane worldsheet projection operator  $Z$ , the same definition yields projection gradients associated with the complementary projection tensor  $V$ :

$$\nabla_{\check{\beta}} V^\alpha{}_{\check{\gamma}} := v^\alpha{}_{\check{\gamma}}{}_{\check{\beta}} = v^{\hat{\alpha}}{}_{\check{\gamma}}{}_{\check{\beta}}, \tag{9}$$

$$\nabla_{\hat{\beta}} V^{\check{\alpha}}{}_\gamma := v_{\check{\gamma}}{}^{\check{\alpha}}{}_{\hat{\beta}} = v_{\check{\gamma}}{}^{\hat{\alpha}}{}_{\hat{\beta}}. \tag{10}$$

Now we can use the decomposition of the identity tensor  $I = Z + V$  to force a decomposition of the projection gradient w.r.t. the brane worldsheet:

$$\nabla_\delta Z^\alpha{}_\gamma = \nabla_{\hat{\delta}} Z^{\hat{\alpha}}{}_{\check{\gamma}} + \nabla_{\check{\delta}} Z^{\check{\alpha}}{}_{\check{\gamma}} - \nabla_{\check{\delta}} V^{\hat{\alpha}}{}_{\check{\gamma}} - \nabla_{\hat{\delta}} V^{\check{\alpha}}{}_{\check{\gamma}} \tag{11}$$

or equivalently

$$\nabla_\delta Z^\alpha{}_\gamma = z_{\check{\gamma}}{}^{\hat{\alpha}}{}_{\hat{\delta}} + z^{\check{\alpha}}{}_{\check{\gamma}}{}_{\hat{\delta}} - v_{\check{\gamma}}{}^{\hat{\alpha}}{}_{\check{\delta}} - v_{\check{\gamma}}{}^{\check{\alpha}}{}_{\check{\delta}}. \tag{12}$$

The decomposition of the complementary projection gradient  $\nabla V$  is given by the complement of this expression, namely by exchanging  $Z$  and  $V$ , as well as  $\wedge$  and  $\vee$  everywhere.

Because each projection gradient has two indices which project into the same subspace, we can contract these two indices and thus extract symmetric and antisymmetric parts. From them we can define the kinematical quantities associated with a projection tensor field.

The antisymmetric part reads

$$\nabla_{[\delta} Z^{\check{\alpha}}_{\check{\gamma}]} := \hat{\omega}_{\gamma\delta}{}^{\alpha} \tag{13}$$

and we call it the projection vorticity tensor.

The symmetric part reads

$$\nabla_{(\delta} Z^{\check{\alpha}}_{\check{\gamma})} := \hat{\theta}_{\gamma\delta}{}^{\alpha} \tag{14}$$

and we call it the projection expansion rate tensor.

When a metric tensor is available for raising and lowering indices we can also define a trace part

$$\nabla_{(\delta} Z^{\check{\gamma}}_{\check{\delta})} := \hat{\theta}^{\gamma}{}_{\gamma}, \tag{15}$$

which we call the projection divergence, as well as a trace-free symmetric part

$$\hat{\sigma}_{\gamma\delta}{}^{\alpha} := \hat{\theta}_{\gamma\delta}{}^{\alpha} - \frac{1}{p} Z_{\gamma\delta} \hat{\theta}^{\alpha}{}_{\alpha}, \tag{16}$$

which we call the projection shear tensor.

### 2.3. Intrinsic and extrinsic brane worldsheet projected covariant derivatives

We consider a vector field  $u$  which obeys the projection identity  $Zu = u$ , meaning that  $u(P) \in ZT_P$  for every point  $P$  in the background manifold, namely the tangent space of the brane worldsheet. The part of the covariant derivative  $\nabla u$  reflecting the way in which the vector field  $u$  changes within the tangent space of the brane worldsheet is defined to be the brane worldsheet projected covariant derivative  $Du$  with components

$$D_{\delta} u^{\hat{\alpha}} = Z^{\alpha}_{\rho} \nabla_{\delta} u^{\hat{\rho}}. \tag{17}$$

On the other hand the part of the covariant derivative ignoring the way in which  $u$  changes within the tangent space of the brane worldsheet is defined to be the brane worldsheet antiprojected covariant derivative with components

$$\tilde{D}_{\delta} u^{\hat{\alpha}} = V^{\alpha}_{\rho} \nabla_{\delta} u^{\hat{\rho}}. \tag{18}$$

We can easily show that the derivatives defined above give zero when they act on the projection tensor fields themselves:

$$D_{\delta} Z^{\alpha}_{\beta} = 0, \quad D_{\delta} V^{\alpha}_{\beta} = 0. \tag{19}$$

It is to be noted that these derivatives act only on tensor fields belonging to particular projected subspaces and thus obey the product rule only for products of these tensors.

The projected and antiprojected covariant derivatives are not entirely projected geometrical objects. The desired entirely projected objects are the following:

$$D_{\hat{\delta}} u^{\hat{\alpha}} = Z^{\rho}_{\delta} D_{\rho} u^{\hat{\alpha}}, \tag{20}$$

which we call the intrinsic brane worldsheet projected covariant derivative;

$$D_{\check{\delta}} u^{\hat{\alpha}} = V^{\rho}_{\delta} D_{\rho} u^{\hat{\alpha}}, \tag{21}$$

which we call the extrinsic brane worldsheet projected covariant derivative;

$$\tilde{D}_{\hat{\delta}} u^{\hat{\alpha}} = Z^{\rho}_{\delta} \tilde{D}_{\rho} u^{\hat{\alpha}}, \tag{22}$$

which we call the intrinsic brane worldsheet antiprojected covariant derivative; and finally

$$\tilde{D}_{\check{\delta}} u^{\hat{\alpha}} = V^{\rho}_{\delta} \tilde{D}_{\rho} u^{\hat{\alpha}}, \tag{23}$$

which we call the extrinsic brane worldsheet antiprojected covariant derivative.

Using the geometrical objects notation we find that the full decomposition of the background manifold connection w.r.t. the brane worldsheet into entirely projected parts is determined by

$${}^{\rho} \nabla_{(\delta)} u^{(\alpha)} = D_{(\delta)} u^{(\alpha)} + u^{[\sigma]} \mathcal{E}^{(\alpha)}_{[\sigma](\delta)}, \tag{24}$$

where

$$\mathcal{E}^{(\alpha)}_{(\sigma)(\delta)} := \nabla Z^{(\alpha)}_{[\mu](\delta)} Y^{[\mu]}_{(\sigma)} \tag{25}$$

and

$$Y^{\sigma}_{\rho} := Z^{\sigma}_{\rho} - V^{\sigma}_{\rho}. \tag{26}$$

In the above formula the geometrical object  $\mathcal{E}^{(\alpha)}_{(\sigma)(\delta)}$  does not change sign under complementation and stands as the generator of correction terms in the relationship between the covariant and the entirely projected w.r.t. the brane worldsheet derivatives. If we compute the components of  $\mathcal{E}^{(\alpha)}_{(\sigma)(\delta)}$ , we obtain

$$\begin{aligned} \mathcal{E}^{\hat{\beta}}_{\hat{\alpha}\hat{\delta}} &= 0, & \mathcal{E}^{\hat{\beta}}_{\check{\alpha}\hat{\delta}} &= -z^{\hat{\beta}}_{\check{\alpha}\hat{\delta}}, \\ \mathcal{E}^{\check{\beta}}_{\hat{\alpha}\hat{\delta}} &= z^{\check{\beta}}_{\hat{\alpha}\hat{\delta}}, & \mathcal{E}^{\check{\beta}}_{\check{\alpha}\hat{\delta}} &= 0, \\ \mathcal{E}^{\hat{\beta}}_{\hat{\alpha}\check{\delta}} &= 0, & \mathcal{E}^{\hat{\beta}}_{\check{\alpha}\check{\delta}} &= v^{\hat{\beta}}_{\check{\alpha}\check{\delta}}, \\ \mathcal{E}^{\check{\beta}}_{\hat{\alpha}\check{\delta}} &= -v^{\check{\beta}}_{\hat{\alpha}\check{\delta}}, & \mathcal{E}^{\check{\beta}}_{\check{\alpha}\check{\delta}} &= 0. \end{aligned}$$

#### 2.4. Decomposition of the metricity and torsion w.r.t. the brane worldsheet

We consider that the background manifold is endowed with a form-metric  $g^{\mu\nu}$ . The metricity tensor is defined by the relation

$$Q^{\mu\nu}_{\rho} = -\nabla_{\rho} g^{\mu\nu}. \tag{27}$$

In turn we define the entirely projected metricity geometrical object w.r.t. the brane worldsheet by the relation

$$Q^{(\mu)(\nu)}_{(\rho)} := -D_{(\rho)}g^{(\mu)(\nu)}. \tag{28}$$

In the above definition the entirely projected geometrical object  $Q^{(\mu)(\nu)}_{(\rho)}$  includes the intrinsic metricity  $Q^{\hat{\mu}\hat{\nu}}_{\hat{\rho}}$  associated with the subspace  $ZT_P$ , the intrinsic metricity  $Q^{\check{\mu}\check{\nu}}_{\check{\rho}}$  associated with the subspace  $VT_P$ , as well as the mixed projected metricities.

Using the geometrical objects notation we find that the decomposition of the full background manifold metricity w.r.t. the brane worldsheet into entirely projected parts is determined by the formula:

$${}^pQ^{(\mu)(\nu)}_{(\delta)} = Q^{(\mu)(\nu)}_{(\delta)} - 2g^{[\rho][(\mu)}\Xi^{(\nu)]}_{[\rho](\delta)}. \tag{29}$$

Formula (29) is equivalent to the following system of equations in the projected index formalism:

$${}^pQ^{\hat{\mu}\hat{\nu}}_{\hat{\delta}} = Q^{\hat{\mu}\hat{\nu}}_{\hat{\delta}} + g^{\hat{\mu}\hat{\rho}}z_{\hat{\rho}\hat{\delta}}^{\hat{\nu}} + g^{\check{\rho}\check{\nu}}z_{\hat{\rho}\hat{\delta}}^{\hat{\mu}}, \tag{30}$$

$${}^pQ^{\hat{\mu}\check{\nu}}_{\hat{\delta}} = Q^{\hat{\mu}\check{\nu}}_{\hat{\delta}} - g^{\hat{\mu}\hat{\rho}}v_{\hat{\rho}\check{\delta}}^{\check{\nu}} - g^{\check{\rho}\check{\nu}}v_{\hat{\rho}\check{\delta}}^{\hat{\mu}}, \tag{31}$$

$${}^pQ^{\check{\mu}\check{\nu}}_{\check{\delta}} = Q^{\check{\mu}\check{\nu}}_{\check{\delta}} - g^{\hat{\mu}\hat{\rho}}z_{\hat{\rho}\check{\delta}}^{\check{\nu}} + g^{\check{\rho}\check{\nu}}z_{\hat{\rho}\check{\delta}}^{\hat{\mu}}, \tag{32}$$

$${}^pQ^{\hat{\mu}\check{\nu}}_{\check{\delta}} = Q^{\hat{\mu}\check{\nu}}_{\check{\delta}} + g^{\hat{\mu}\hat{\rho}}v_{\hat{\rho}\check{\delta}}^{\check{\nu}} - g^{\check{\rho}\check{\nu}}v_{\hat{\rho}\check{\delta}}^{\hat{\mu}} \tag{33}$$

and their complements.

We note that if the background manifold is endowed with a regular tensor and the connection is metric compatible, the metricity tensor vanishes, and thus the left-hand sides of the above equations equal zero. Moreover, for a normal projection tensor field the mixed projected metrics vanish. Finally, we observe that the intrinsic and the mixed projected metricities do not necessarily vanish for a non-normal projection tensor field even if the connection is metric compatible.

The torsion tensor is defined by the relation

$$[\nabla_\nu, \nabla_\mu]\phi = S^\rho_{\mu\nu}\nabla_\rho\phi. \tag{34}$$

The above definition can be decomposed into entirely projected parts w.r.t. the brane worldsheet by defining the entirely projected torsion tensors according to

$$[D_{(\nu)}, D_{(\mu)}]\phi := S^{(\rho)}_{(\mu)(\nu)}D_{(\rho)}\phi. \tag{35}$$

Since  $Z$  is the projection tensor onto the brane worldsheet the quantity  $[D_{\hat{\nu}}, D_{\hat{\mu}}]\phi$  is related to the torsion of the intrinsic geometry on this surface.

In definition (35) the entirely projected geometrical object  $S^{(\rho)}_{(\mu)(\nu)}$  includes the intrinsic torsion  $S^{\hat{\rho}}_{\hat{\mu}\hat{\nu}}$  associated with the subspace  $ZT_P$ , the intrinsic torsion  $S^{\check{\rho}}_{\check{\mu}\check{\nu}}$  associated with the subspace  $VT_P$ , as well as the mixed projected torsions.

Using the geometrical objects notation we find that the decomposition of the full background manifold torsion w.r.t. the brane worldsheet into entirely projected parts is determined by the formula:

$${}^p S^{(\rho)}_{(\mu)(\nu)} = S^{(\rho)}_{(\mu)(\nu)} - 2\mathcal{E}^{(\rho)}_{[(\mu)(\nu)]}. \tag{36}$$

Formula (36) is equivalent to the following system of equations in the projected index formalism:

$${}^p S^{\hat{\rho}}_{\hat{\mu}\hat{\nu}} = S^{\hat{\rho}}_{\hat{\mu}\hat{\nu}}, \tag{37}$$

$${}^p S^{\check{\rho}}_{\hat{\mu}\hat{\nu}} = S^{\check{\rho}}_{\hat{\mu}\hat{\nu}} - 2\hat{\omega}^{\rho}_{\mu\nu}, \tag{38}$$

$${}^p S^{\hat{\rho}}_{\hat{\mu}\check{\nu}} = S^{\hat{\rho}}_{\hat{\mu}\check{\nu}} - z^{\hat{\rho}}_{\check{\nu}} \hat{\mu}. \tag{39}$$

together with their complements.

Furthermore, if there is a regular metric tensor, and the background manifold connection is torsion free we obtain

$${}^p S^{\hat{\rho}}_{\hat{\mu}\hat{\nu}} = {}^p S^{\check{\rho}}_{\hat{\mu}\hat{\nu}} = {}^p S^{\check{\rho}}_{\hat{\mu}\check{\nu}} = 0 \tag{40}$$

or equivalently

$$S^{\hat{\rho}}_{\hat{\mu}\hat{\nu}} = 0, \tag{41}$$

$$S^{\check{\rho}}_{\hat{\mu}\hat{\nu}} = 2\hat{\omega}^{\rho}_{\mu\nu}, \tag{42}$$

$$S^{\hat{\rho}}_{\hat{\mu}\check{\nu}} = z^{\hat{\rho}}_{\check{\nu}} \hat{\mu}. \tag{43}$$

The  $\wedge\wedge$ -projection of the torsion definition reads

$$[D_{\hat{\nu}}, D_{\hat{\mu}}]\phi = 2\hat{\omega}^{\rho}_{\mu\nu} D_{\hat{\rho}}\phi + S^{\rho}_{\hat{\mu}\hat{\nu}} \nabla_{\rho}\phi. \tag{44}$$

In an adapted frame the  $\wedge\wedge$ -projection of the torsion definition takes the form

$$([e_n, e_m] - (2\Gamma^r_{[mn]} + \Sigma^r_{mn})e_r - \Sigma^R_{mn}e_R)\phi = 0. \tag{45}$$

The above expression provides the needed relation with the Frobenius theorem. Concretely, it shows that two vector fields with values in the subspace  $ZT_P$  have a commutator which lays in the same subspace if and only if the mixed projected torsion tensor  $\Sigma^R_{mn}$  is zero. We call this condition generalised Weingarden identity. Then the Frobenius theorem guarantees that the subspace  $ZT_P$  is tangent to a submanifold which is identified as the brane worldsheet. As a consequence, if the background manifold connection is torsion free, the vanishing of the mixed projected tensor  $\Sigma^R_{mn}$  implies, according to (42), that the vorticity tensor  $\hat{\omega}^{\rho}_{\mu\nu}$  should vanish.

### 2.5. Decomposition of the Riemann curvature w.r.t. the brane worldsheet

It is possible to relate the geometry of the projection tensor field onto the brane worldsheet with the Riemannian geometry that it inhabits, by decomposing the space-time Riemann tensor w.r.t. the projection.

The curvature tensor is defined by the equation

$$u^\rho R_\rho{}^\gamma{}_{\alpha\beta} = ([\nabla_\beta, \nabla_\alpha] - S^\rho{}_{\alpha\beta} \nabla_\rho) u^\gamma. \tag{46}$$

We can decompose the above definition, and from all of its independent projections. In order to do this we first define the entirely projected curvature geometrical object by the action of the entirely projected covariant derivatives on the entirely projected w.r.t. the brane worldsheet vector fields according to

$$u^{[\rho]} R_{[\rho]}^{(\gamma)}{}_{(\alpha)(\beta)} := ([D_{(\beta)}, D_{(\alpha)}] - S^{[\rho]}{}_{(\alpha)(\beta)} D_{[\rho]}) u^{(\gamma)}. \tag{47}$$

In the above definition the first two pairs of indices of the entirely projected curvature geometrical object must belong to the same projected subspace. So we demand that

$$R_{\hat{\rho}}^{\check{\gamma}}{}_{(\alpha)(\beta)} = R_{\check{\rho}}^{\hat{\gamma}}{}_{(\alpha)(\beta)} = 0.$$

Since the projection tensor  $Z$  is surface forming it is clear that  $R_{\hat{\rho}}^{\hat{\gamma}}{}_{\hat{\alpha}\hat{\beta}}$  is the intrinsic curvature tensor of the brane worldsheet. In an adapted frame the components of  $R_{\hat{\rho}}^{\hat{\gamma}}{}_{\hat{\alpha}\hat{\beta}}$  are given by the expression

$$R_{r\ ab}^c = e_b(\Gamma^c{}_{ra}) - e_a(\Gamma^c{}_{rb}) + \Gamma^s{}_{ra} \Gamma^c{}_{sb} - \Gamma^s{}_{rb} \Gamma^c{}_{sa} - \Gamma^c{}_{rs}(2\Gamma^s{}_{[ab]} - \Sigma^s{}_{ab}) - \Sigma^s{}_{ab} \Gamma^c{}_{rs}. \tag{48}$$

Similarly we call  $R_{\check{\rho}}^{\check{\gamma}}{}_{\check{\alpha}\check{\beta}}$  the intrinsic curvature associated with the projection tensor field  $V$ , whereas we call  $R_{\hat{\rho}}^{\check{\gamma}}{}_{\hat{\alpha}\check{\beta}}$  mixed projected curvature tensor.

Now it is possible to express the complete decomposition of the background manifold Riemannian curvature tensor w.r.t. the brane worldsheet in terms of the intrinsic and mixed projected curvature tensors. Using the geometrical objects, notation the decomposition of the full background manifold Riemann tensor w.r.t. the brane worldsheet into entirely projected parts is determined by the formula:

$${}^p R_{(\rho)}^{(\gamma)}{}_{(\alpha)(\beta)} = R_{(\rho)}^{(\gamma)}{}_{(\alpha)(\beta)} + 2D_{[(\beta)} \Xi^{(\gamma)}{}_{|(\rho)||(\alpha)]} - S^{[\sigma]}{}_{(\alpha)(\beta)} \Xi^{(\gamma)}{}_{(\rho)[\sigma]} - 2\Xi^{(\gamma)}{}_{[\delta][(\alpha)} \Xi^{[\delta]}{}_{|(\rho)||(\beta)]}. \tag{49}$$

Formula (49) is equivalent to the following system of equations in the period index formalism:

$${}^p R_{\hat{\rho}}^{\hat{\gamma}}{}_{\hat{\alpha}\hat{\beta}} = R_{\hat{\rho}}^{\hat{\gamma}}{}_{\hat{\alpha}\hat{\beta}} - z_{\check{\sigma}}^{\hat{\gamma}}{}_{\hat{\beta}} z^{\check{\sigma}}{}_{\hat{\rho}\hat{\alpha}} + z_{\check{\sigma}}^{\hat{\gamma}}{}_{\hat{\alpha}} z^{\check{\sigma}}{}_{\hat{\rho}\hat{\beta}}, \tag{50}$$

$${}^p R_{\hat{\rho}}^{\hat{\gamma}}{}_{\hat{\alpha}\check{\beta}} = R_{\hat{\rho}}^{\hat{\gamma}}{}_{\hat{\alpha}\check{\beta}} + z^{\check{\sigma}}{}_{\hat{\rho}\hat{\alpha}} v^{\hat{\gamma}}{}_{\check{\sigma}\check{\beta}} - z_{\check{\sigma}}^{\hat{\gamma}}{}_{\hat{\alpha}} v_{\check{\rho}}^{\check{\sigma}}{}_{\check{\beta}}, \tag{51}$$

$${}^p R_{\hat{\rho}}^{\hat{\gamma}}{}_{\check{\alpha}\check{\beta}} = R_{\hat{\rho}}^{\hat{\gamma}}{}_{\check{\alpha}\check{\beta}} + v^{\hat{\gamma}}{}_{\check{\sigma}\check{\alpha}} v_{\check{\rho}}^{\check{\sigma}}{}_{\check{\beta}} - v_{\check{\sigma}\check{\beta}}^{\hat{\gamma}} v_{\check{\rho}}^{\check{\sigma}}{}_{\check{\alpha}}, \tag{52}$$

$${}^p R_{\hat{\rho}}^{\check{\gamma}}{}_{\hat{\alpha}\hat{\beta}} = D_{\hat{\beta}} z^{\check{\gamma}}{}_{\hat{\rho}\hat{\beta}} - D_{\hat{\alpha}} z^{\check{\gamma}}{}_{\hat{\rho}\hat{\beta}} - z_{\hat{\rho}\hat{\sigma}}^{\check{\gamma}} S_{\hat{\alpha}\hat{\beta}}^{\hat{\sigma}} + v_{\hat{\rho}}^{\check{\gamma}}{}_{\check{\sigma}} S_{\hat{\alpha}\hat{\beta}}^{\hat{\sigma}}, \tag{53}$$

$${}^p R_{\hat{\rho}}^{\check{\gamma}}{}_{\hat{\alpha}\check{\beta}} = D_{\check{\beta}} z^{\check{\gamma}}{}_{\hat{\rho}\hat{\alpha}} - D_{\hat{\alpha}} v_{\check{\rho}}^{\check{\gamma}}{}_{\check{\beta}} - z_{\hat{\rho}\hat{\sigma}}^{\check{\gamma}} S_{\hat{\alpha}\check{\beta}}^{\hat{\sigma}} + v_{\hat{\rho}}^{\check{\gamma}}{}_{\check{\sigma}} S_{\hat{\alpha}\check{\beta}}^{\hat{\sigma}}, \tag{54}$$

$${}^p R_{\hat{\rho}}{}^{\check{\gamma}}{}_{\check{\alpha}\check{\beta}} = -D_{\check{\beta}} v_{\hat{\rho}}{}^{\check{\gamma}}{}_{\check{\alpha}} + D_{\check{\alpha}} v_{\hat{\rho}}{}^{\check{\gamma}}{}_{\check{\beta}} - z^{\check{\gamma}}{}_{\hat{\rho}\hat{\sigma}} S_{\check{\alpha}\check{\beta}}^{\hat{\sigma}} + v_{\hat{\rho}}{}^{\check{\gamma}}{}_{\hat{\sigma}} S_{\check{\alpha}\check{\beta}}^{\hat{\sigma}} \tag{55}$$

and their complements.

We note that Eqs. (50) and (53) are the generalisations of the Gauss–Codazzi relations. From Eq. (54) we can extract an integrability condition in the following form:

$$R_{[\hat{\rho}}{}^{\check{\beta}}{}_{\check{\alpha}]\check{\beta}} = D_{\check{\beta}} \hat{\omega}^{\beta}{}_{\rho\alpha} + D_{[\check{\alpha}} \check{\theta}_{\rho]} - z^{\check{\beta}}{}_{[\hat{\rho}\hat{\sigma}]} S_{\check{\alpha}]\check{\beta}}^{\hat{\sigma}} + v_{[\hat{\rho}\hat{\sigma}]}^{\check{\gamma}} S_{\check{\alpha}]\check{\beta}}^{\hat{\sigma}} = 0. \tag{56}$$

When the background manifold is endowed with a regular metric tensor, the projection tensor is normal, and the background manifold connection is metric compatible and torsion free, we have only three independent projections of the Riemannian tensor. The three independent projections together with symmetries of the Riemann tensor as well as their complements provide a complete decomposition of the Riemannian curvature.

Finally, it is of interest to consider the decomposition of the contracted curvature, namely the Ricci tensor, w.r.t. the brane worldsheet.

First we construct the  $\wedge$ -projection of the Ricci curvature tensor:

$${}^p R_{\hat{\alpha}\hat{\beta}} = R_{\hat{\alpha}\hat{\beta}} + D_{\hat{\sigma}} z^{\hat{\sigma}}{}_{\hat{\alpha}\hat{\beta}} + D_{\hat{\beta}} \check{\theta}_{\alpha} + z^{\hat{\rho}}{}_{\hat{\alpha}\hat{\sigma}} (z_{\hat{\rho}}{}^{\hat{\sigma}}{}_{\hat{\beta}} - S_{\hat{\rho}\hat{\beta}}^{\hat{\sigma}}) - z^{\hat{\sigma}}{}_{\hat{\alpha}\hat{\beta}} \hat{\theta}_{\sigma} + v_{\hat{\alpha}}{}^{\hat{\rho}}{}_{\hat{\sigma}} S_{\hat{\beta}\hat{\rho}}^{\hat{\sigma}}, \tag{57}$$

where  $R_{\hat{\alpha}\hat{\beta}}$  is the intrinsic Ricci curvature tensor associated with the brane worldsheet.

The  $\vee$ -projection of the Ricci curvature tensor is obtained by taking the complements in (57), and corresponds to the intrinsic Ricci curvature associated with the projection tensor field  $V$ .

By contracting  ${}^p R_{\hat{\alpha}\hat{\beta}}$  with the projected metric tensor  $g^{\hat{\alpha}\hat{\beta}}$  we obtain a generalisation of the Raychaudhuri equation in the form:

$$g^{\hat{\alpha}\hat{\beta}} {}^p R_{\hat{\alpha}\hat{\beta}} = \hat{R} + D_{\hat{\alpha}} \theta^{\hat{\alpha}} + D_{\hat{\beta}} \theta^{\hat{\beta}} - \theta_{\hat{\sigma}} \theta^{\hat{\sigma}} + z_{\hat{\sigma}}{}^{\hat{\rho}\hat{\rho}} z^{\hat{\sigma}}{}_{\hat{\beta}\hat{\rho}} + Q^{\hat{\alpha}\hat{\rho}}{}_{\hat{\alpha}} \check{\theta}_{\rho} + Q^{\hat{\alpha}\hat{\rho}}{}_{\hat{\beta}} z^{\hat{\beta}}{}_{\hat{\rho}\hat{\alpha}} - z^{\hat{\beta}\hat{\alpha}}{}_{\hat{\sigma}} S_{\hat{\alpha}\hat{\beta}}^{\hat{\sigma}} + v_{\hat{\alpha}}{}^{\hat{\beta}}{}_{\hat{\sigma}} S_{\hat{\alpha}\hat{\beta}}^{\hat{\sigma}}. \tag{58}$$

When the background manifold is endowed with a metric compatible and torsion free connection as well as the projection tensor field is normal, Raychaudhuri equation takes the form:

$$g^{\hat{\alpha}\hat{\beta}} {}^p R_{\hat{\alpha}\hat{\beta}} = \hat{R} + D_{\hat{\alpha}} \theta^{\hat{\alpha}} + D_{\hat{\beta}} \theta^{\hat{\beta}} - \theta_{\hat{\sigma}} \theta^{\hat{\sigma}} - v_{\hat{\alpha}}{}^{\hat{\sigma}}{}_{\hat{\beta}} v^{\hat{\alpha}\hat{\beta}}{}_{\hat{\sigma}}. \tag{59}$$

In order to complete our analysis we mention the identities that the entirely projected w.r.t. the brane worldsheet torsion and Riemann curvature geometrical objects obey. These are the following:

1. Entirely projected torsion Bianchi identities:

$$R_{[(\rho)}{}^{(\gamma)}{}_{(\alpha)(\beta)]} + D_{[(\rho)} S^{(\gamma)}{}_{(\alpha)(\beta)]} + S^{(\gamma)}{}_{[\sigma][(\rho)} S^{[\sigma]}{}_{(\alpha)(\beta)} = 0. \tag{60}$$

2. Entirely projected curvature Bianchi identities:

$$D_{[(\rho)} R_{(\delta)]}{}^{(\gamma)}{}_{(\alpha)(\beta)} + R_{(\delta)}{}^{(\gamma)}{}_{[\sigma][(\rho)} S^{[\sigma]}{}_{(\alpha)(\beta)} = 0. \tag{61}$$

### 3. Decomposition of Lie derivatives w.r.t. the brane worldsheet

The Lie derivative of a vector field  $u$  w.r.t. a vector field  $M$  can be obtained by the commutator

$$(\mathcal{L}_M u)\phi = ([M, u])\phi \tag{62}$$

for every scalar field  $\phi$  defined on the background manifold.

The relationship between the Lie derivative and the covariant derivative is provided by the formula

$$(\mathcal{L}_M u)^\delta = M^\sigma \nabla_\sigma u^\delta - u^\rho (\nabla_\rho M^\delta - S^\delta_{\rho\sigma} M^\sigma). \tag{63}$$

It is interesting to work out the brane worldsheet projection tensor decomposition of the Lie derivative of the projection tensor  $Z$ .

**Proposition 1.** *The full decomposition of the background manifold Lie derivative of the projection tensor  $Z$  w.r.t the brane worldsheet into entirely projected parts is determined by the formula:*

$$\begin{aligned} \mathcal{L}_M Z^{(\alpha)}_{(\sigma)} &= \frac{1}{2} [Y^{(\alpha)}_{[\beta]} D_{(\sigma)} M^{[\beta]} - Y^{[\beta]}_{(\sigma)} D_{[\beta]} M^{(\alpha)}] \\ &\quad + \frac{1}{2} M^{[\delta]} [Y^{[\beta]}_{(\sigma)} S^{(\alpha)}_{[\beta][\delta]} - Y^{(\alpha)}_{[\beta]} S^{[\beta]}_{(\sigma)[\delta]}]. \end{aligned} \tag{64}$$

*Proof.* The relation between the Lie derivative of the projection tensor  $Z$  with its covariant derivative is given by

$$\mathcal{L}_M Z^\alpha_\beta = M^\delta \nabla_\delta Z^\alpha_\beta - Z^\rho_\beta {}^s \nabla_\rho M^\alpha + Z^\alpha_\rho {}^s \nabla_\beta M^\rho, \tag{65}$$

where

$${}^s \nabla_\rho M^\delta = \nabla_\rho M^\delta - S^\delta_{\rho\sigma} M^\sigma. \tag{66}$$

We consider the projection of the above relation w.r.t. the brane worldsheet, using the geometrical objects notation to obtain

$$\mathcal{L}_M Z^{(\alpha)}_{(\beta)} = M^{[\delta]} {}^p \nabla_{[\delta]} Z^{(\alpha)}_{(\beta)} - Z^{[\rho]}_{(\beta)} {}^{ps} \nabla_{[\rho]} M^{(\alpha)} + Z^{(\alpha)}_{[\rho]} {}^{ps} \nabla_{(\beta)} M^{[\rho]}. \tag{67}$$

If we use the relation between the covariant derivative and the entirely projected covariant derivative as well as the proposition referring to the torsion tensor, we can calculate the r.h.s. of (67) in terms of entirely projected geometrical objects as follows:

$$\begin{aligned} {}^p \mathcal{L}_M Z^{(\alpha)}_{(\beta)} &= Z^{(\alpha)}_{[\rho]} D_{(\beta)} M^{[\rho]} - Z^{[\rho]}_{(\beta)} D_{[\rho]} M^{(\alpha)} \\ &\quad + M^{[\sigma]} [Z^{[\rho]}_{(\beta)} S^{(\alpha)}_{[\rho][\sigma]} - Z^{(\alpha)}_{[\rho]} S^{[\rho]}_{(\beta)[\sigma]}]. \end{aligned} \tag{68}$$

Next we note that the following identity holds:

$$\mathcal{L}_M (Z + V) = \mathcal{L}_M Z + \mathcal{L}_M V = 0. \tag{69}$$

Eq. (69) implies that (68) is antisymmetric under complementation. This property becomes manifest if we make use of the tensor  $Y^\alpha_\beta$  defined by (26), and thus write (68) equivalently in the form:

$$\begin{aligned} \mathcal{L}_M Z^{(\alpha)}_{(\sigma)} &= \frac{1}{2} [Y^{(\alpha)}_{[\beta]} D_{(\sigma)} M^{[\beta]} - Y^{[\beta]}_{(\sigma)} D_{[\beta]} M^{(\alpha)}] \\ &\quad + \frac{1}{2} M^{[\delta]} [Y^{[\beta]}_{(\sigma)} S^{(\alpha)}_{[\beta][\delta]} - Y^{(\alpha)}_{[\beta]} S^{[\beta]}_{(\sigma)[\delta]}], \end{aligned} \tag{70}$$

which completes the proof of the proposition. □

If we assume that the background manifold connection is torsion free, then the only non-zero projections of the Lie derivative of the projection tensor  $Z$  w.r.t. the brane worldsheet, in the projected index notation, are given by

$$(\mathcal{L}_M Z)^{\hat{\alpha}}_{\check{\beta}} = -(S_M)^{\hat{\alpha}}_{\check{\beta}} + D_{\check{\beta}} M^{\hat{\alpha}} + z_{\check{\beta}}^{\hat{\alpha}} M^{\hat{\sigma}}_{\check{\sigma}} - 2\hat{\omega}^{\alpha}_{\beta\sigma} M^{\check{\sigma}}, \tag{71}$$

$$(\mathcal{L}_M Z)^{\check{\alpha}}_{\hat{\beta}} = -(S_M)^{\check{\alpha}}_{\hat{\beta}} + D_{\hat{\beta}} M^{\check{\alpha}} - v_{\hat{\beta}}^{\check{\alpha}} M^{\check{\sigma}} + 2\hat{\omega}^{\alpha}_{\beta\sigma} M^{\hat{\sigma}}, \tag{72}$$

where

$$(S_M)^{\check{\alpha}}_{\hat{\beta}} := S^{\check{\alpha}}_{\hat{\beta}\hat{\sigma}} M^{\hat{\sigma}} + S^{\check{\alpha}}_{\hat{\beta}\check{\sigma}} M^{\check{\sigma}}.$$

We consider a vector field  $u$  which obeys the projection identity  $Zu = u$ . According to the definitions provided in the case of covariant derivatives we define its brane worldsheet entirely projected Lie derivative by the formula

$$L_M u^{\hat{\alpha}} = Z^\alpha_\rho \mathcal{L}_M u^{\hat{\rho}}, \tag{73}$$

whereas its brane worldsheet entirely antiprojected Lie derivative by the formula

$$\tilde{L}_M u^{\hat{\alpha}} = V^\alpha_\rho \mathcal{L}_M u^{\hat{\rho}}. \tag{74}$$

It is important to relate the brane worldsheet entirely tangentially projected Lie derivative to the ordinary Lie derivative.

**Proposition 2.** *The full decomposition of the background manifold Lie derivative of an entirely projected vector field w.r.t. the brane worldsheet into entirely projected parts is determined by*

$$(\mathcal{L}_M^p u)^{(\alpha)} = L_M u^{(\alpha)} - u^{[\rho]} \mathcal{E}_M^{(\alpha)}_{[\rho]}, \tag{75}$$

where

$$\begin{aligned} \mathcal{E}_M^{(\alpha)}_{(\rho)} &= \frac{1}{2} [Y^{[\beta]}_{[\sigma]} D_{[\beta]} M^{(\alpha)} - Y^{(\alpha)}_{[\beta]} D_{[\sigma]} M^{[\beta]}] Y^{[\sigma]}_{(\rho)} \\ &\quad + \frac{1}{2} M^{[\gamma]} [Y^{(\alpha)}_{[\beta]} S^{[\beta]}_{[\sigma][\gamma]} - Y^{[\beta]}_{[\sigma]} S^{(\alpha)}_{[\beta][\gamma]}] Y^{[\sigma]}_{(\rho)} \end{aligned} \tag{76}$$

*Proof.* We start with the expressions:

$$u^{(\alpha)} = Z^{(\alpha)}_{[\rho]} u^{[\rho]} \quad \text{if } u \in ZTP,$$

$$u^{(\alpha)} = V^{(\alpha)}_{[\rho]} u^{[\rho]} \quad \text{if } u \in VTP.$$

We Lie-differentiate the above expressions:

$$\mathcal{L}_M u^{(\alpha)} = (\mathcal{L}_m Z^{(\alpha)}_{[\rho]} u^{[\rho]} + Z^{(\alpha)}_{[\rho]} \mathcal{L}_M u^{[\rho]}), \tag{77}$$

$$\mathcal{L}_M u^{(\alpha)} = (\mathcal{L}_m Z^{(\alpha)}_{[\rho]} u^{[\rho]} + V^{(\alpha)}_{[\rho]} \mathcal{L}_M u^{[\rho]}) \tag{78}$$

for  $u \in ZTP$  or  $VTP$ , respectively.

If we make use of the tensor  $Y^\alpha_\beta$ , Eqs. (77) and (78) can be written in a single relation as follows:

$$\mathcal{L}_M u^{(\alpha)} = (\mathcal{L}_M Z^{(\alpha)}_{[\sigma]} Y^{[\sigma]}_{[\rho]} u^{[\rho]} + L_M u^{(\alpha)}), \tag{79}$$

where the following identity is satisfied:

$$-(\mathcal{L}_M Z^{(\alpha)}_{[\sigma]}) Y^{[\sigma]}_{(\beta)} = Y^{(\alpha)}_{[\sigma]} (\mathcal{L}_M Z^{[\sigma]}_{(\beta)}). \tag{80}$$

By definition we have

$$\begin{aligned} \mathfrak{E}_M^{(\alpha)}_{(\rho)} &= \frac{1}{2} [Y^{[\beta]}_{[\sigma]} D_{[\beta]} M^{(\alpha)} - Y^{(\alpha)}_{[\beta]} D_{[\sigma]} M^{[\beta]}] Y^{[\sigma]}_{(\rho)} \\ &\quad + \frac{1}{2} M^{(\gamma)} [Y^{(\alpha)}_{[\beta]} S^{[\beta]}_{[\sigma][\gamma]} - Y^{[\beta]}_{[\sigma]} S^{(\alpha)}_{[\beta][\gamma]}] Y^{[\sigma]}_{(\rho)}. \end{aligned} \tag{81}$$

Hence if we combine (70), (79) and (81), we finally obtain

$$(\mathcal{L}_M^p u)^{(\alpha)} = L_M u^{(\alpha)} - u^{[\rho]} \mathfrak{E}_M^{(\alpha)}_{[\rho]}, \tag{82}$$

which completes the proof of the proposition. □

Then Eq. (82), after using (71) and (72) in the projected index notation takes the form:

$$\begin{aligned} \mathcal{L}_M u^{\hat{\alpha}} &= L_M u^{\hat{\alpha}} - (S_M)^{\hat{\alpha}}_{\hat{\beta}} + D_{\hat{\beta}} M^{\hat{\alpha}} + z^{\hat{\alpha}}_{\hat{\beta}} M^{\hat{\alpha}} - \hat{\omega}^{\alpha}_{\beta\sigma} M^{\hat{\sigma}} \\ &\quad + (S_M)^{\hat{\alpha}}_{\hat{\beta}} - D_{\hat{\beta}} M^{\hat{\alpha}} - v^{\hat{\alpha}}_{\hat{\beta}} M^{\hat{\sigma}} + \hat{\omega}^{\alpha}_{\beta\sigma} M^{\hat{\sigma}}) u^{\hat{\beta}}. \end{aligned} \tag{83}$$

Furthermore, we can generalise the above notions and ask for the decomposition of the Lie derivative of a general entirely projected tensor w.r.t. the brane worldsheet. So let us consider the entirely projected geometrical object  $X^{(\alpha)(\beta)}_{(\mu)(\nu)}$ :

$$\begin{aligned} (\mathcal{L}_M X)^{(\alpha)(\beta)}_{(\mu)(\nu)} &= L_M X^{(\alpha)(\beta)}_{(\mu)(\nu)} - X^{[\rho](\beta)}_{(\mu)(\nu)} \mathfrak{E}_M^{(\alpha)}_{[\rho]} \\ &\quad - X^{(\alpha)[\rho]}_{(\mu)(\nu)} \mathfrak{E}_M^{(\beta)}_{[\rho]} + X^{(\alpha)(\beta)}_{[\rho](\nu)} \mathfrak{E}_M^{[\rho]}_{(\mu)} \\ &\quad + X^{(\alpha)(\beta)}_{(\mu)[\rho]} \mathfrak{E}_M^{[\rho]}_{(\nu)}. \end{aligned} \tag{84}$$

A significant application of the above decompositions is obtained when we study the evolution of a timelike brane worldsheet area element. Let us suppose that  $\dim ZTP = p$ .

The area element on the brane worldsheet surface tangent to the subspace  $ZT_P$  is the unit  $p$ -form  $Q$  in  $Z^*T_P$ . We choose a vector field  $N^a$  in  $VT_P$ , and a  $p$ -form field  $S$  which is propagated along the integral curves of  $N^a$  by Lie-dragging so that it is a solution of

$$\mathcal{L}_N S = 0.$$

So long as the choosen Lie-dragged  $p$ -form  $S$  obeys

$$Z^* S \neq 0. \tag{85}$$

The  $Z$ -area element  $Q$  can be constructed from  $S$ . Since the subspace  $ZT_P$  is timelike we have

$$Q = (\det[g_{\hat{\alpha}\hat{\beta}}])^{1/2} Z^* S, \tag{86}$$

where

$$(\det[g_{\hat{\alpha}\hat{\beta}}]) \neq 0. \tag{87}$$

Because the entirely projected Lie derivative obeys

$$L_N Z = 0, \tag{88}$$

and the field  $S$  has been defined by Lie-dragging, we see that

$$L_N (Z^* S) = 0. \tag{89}$$

Then the entirely projected Lie derivative of the tangent to the brane worldsheet  $Z$ -area element is

$$L_N Q = -\frac{1}{2} g_{\hat{\alpha}\hat{\beta}} L_N g^{\hat{\alpha}\hat{\beta}} Q. \tag{90}$$

Furthermore, we can calculate the quantity  $L_N g^{\hat{\alpha}\hat{\beta}}$  as follows:

$$({}^p \mathcal{L}_N g)^{(\alpha)(\beta)} = L_N g^{(\alpha)(\beta)} - g^{[\rho](\beta)} \mathcal{E}_N^{(\alpha)}{}_{[\rho]} - g^{(\alpha)[\rho]} \mathcal{E}_N^{(\beta)}{}_{[\rho]}. \tag{91}$$

Hence we obtain

$$\begin{aligned} ({}^p \mathcal{L}_N g)^{\hat{\alpha}\hat{\beta}} &= L_N g^{\hat{\alpha}\hat{\beta}} - g^{\hat{\rho}\hat{\beta}} \mathcal{E}_N^{\hat{\alpha}}{}_{\hat{\rho}} - g^{\hat{\rho}\hat{\beta}} \mathcal{E}_N^{\hat{\alpha}}{}_{\hat{\rho}} \\ &\quad - g^{\hat{\alpha}\hat{\rho}} \mathcal{E}_N^{\hat{\beta}}{}_{\hat{\rho}} - g^{\hat{\alpha}\hat{\rho}} \mathcal{E}_N^{\hat{\beta}}{}_{\hat{\rho}}, \end{aligned} \tag{92}$$

where

$$\begin{aligned} \mathcal{E}_N^{\hat{\alpha}}{}_{\hat{\beta}} &= -S_N^{\hat{\alpha}}{}_{\hat{\beta}}, \\ \mathcal{E}_N^{\check{\alpha}}{}_{\hat{\beta}} &= -S_N^{\check{\alpha}}{}_{\hat{\beta}} + D_{\hat{\beta}} N^{\check{\alpha}} + v_{\hat{\beta}}^{\check{\alpha}} N^{\check{\sigma}} - 2\hat{\omega}^{\alpha}{}_{\beta\sigma} M^{\hat{\sigma}} \end{aligned}$$

and the complements of the above expressions.

The geometrical object  $(\mathcal{L}_N^p g)^{(\alpha)(\beta)}$  in terms of the entirely projected covariant derivatives is given by

$$\begin{aligned}
 ({}^P \mathcal{L}_N g)^{(\alpha)(\beta)} &= [2g^{[\rho][\mu]} S^{(v)}_{[\rho][\delta]} - Q^{(\mu)(v)}_{[\delta]}] N^{[\delta]} \\
 &\quad - g^{[\rho](v)} D_{[\rho]} N^{(\mu)} - g^{(\mu)[\rho]} D_{[\rho]} N^{(v)}.
 \end{aligned}
 \tag{93}$$

Now if we use Eqs. (91)–(93) we finally obtain

$$\begin{aligned}
 L_N g^{\hat{\alpha}\hat{\beta}} &= -Q_N^{\hat{\alpha}\hat{\beta}} - 2g^{\hat{\rho}(\hat{\alpha}} D_{\hat{\rho}} N^{\hat{\beta})} + 2g^{\hat{\rho}(\hat{\beta}} z_{\hat{\rho}}^{\hat{\alpha}} N^{\hat{\sigma}} \\
 &\quad + 2(g^{\hat{\rho}(\hat{\alpha}} z_{\hat{\sigma}}^{\hat{\beta})})_{\hat{\rho}} - g^{\hat{\rho}(\hat{\alpha}} v_{\hat{\rho}\hat{\sigma}}^{\hat{\beta})}) N^{\hat{\sigma}},
 \end{aligned}
 \tag{94}$$

where the terms which vanish when we have torsionless and metric compatible connections have been collected in the geometrical object

$$Q^P N^{(\mu)(v)} = (Q^{P(\mu)(v)}_{[\delta]} - g^{(\mu)[\rho]} S^{P(v)}_{[\rho][\delta]} - g^{[\rho](v)} S^{P(\mu)}_{[\rho][\delta]}) N^{[\delta]}.
 \tag{95}$$

Hence, Eq. (90) gives the result

$$g_{\hat{\alpha}\hat{\beta}} (g^{\hat{\rho}(\hat{\alpha}} z_{\hat{\rho}}^{\hat{\beta})})_{\hat{\rho}} - g^{\hat{\rho}(\hat{\alpha}} v_{\hat{\rho}\hat{\sigma}}^{\hat{\beta})}) N^{\hat{\sigma}} Q = -L_N Q.
 \tag{96}$$

Finally, by projecting the identity

$$g_{\alpha\beta} g^{\rho\alpha} = \delta^\rho_\beta
 \tag{97}$$

Eq. (96) takes the form

$$N^\sigma \hat{\theta}_\sigma Q - (g_{\hat{\alpha}\hat{\beta}} g^{\hat{\rho}\hat{\alpha}} z_{\hat{\sigma}}^{\hat{\beta}})_{\hat{\rho}} + g_{\hat{\alpha}\hat{\beta}} g^{\hat{\rho}\hat{\alpha}} v_{\hat{\rho}\hat{\sigma}}^{\hat{\beta}}) N^\sigma Q = -L_N Q.
 \tag{98}$$

Finally, we note that if the projection tensor field onto the brane worldsheet is normal, the above equation is simplified as follows:

$$N^\sigma \hat{\theta}_\sigma Q = -L_N Q.
 \tag{99}$$

Eq. (99) determines the relation between the divergence and the rate of change of the timelike brane worldsheet area element.

#### 4. Summary and discussion

In this paper we have constructed a fully covariant kinematical framework of classical relativistic branes, satisfying needs arising in the context of theories of topological defects structures such as cosmic strings, higher-dimensional cosmic membranes, as well as multi-dimensional fluids. All the previous approaches to such a kinematical description had imposed major restrictions on the kind of brane models considered, excluding any discussion of null branes, as well as of branes in the higher-dimensional manifolds of unified field theories, which use non-metric compatible and non-torsion free connections. Our treatment, based on the use of projection tensor techniques, has removed all the restrictive assumptions. The basic idea we have used is the definition of two different projection gradients which become equal for normal projections.

Our analysis has shown that when a projection tensor field is surface forming, the curvature decomposition includes the generalisation of Gauss–Codazzi equations. When the

projection tensor field is not surface forming, multi-dimensional fluid flow for example, the gradient of the projection tensor turns out to be composed of such well-known quantities as the vorticity and the expansion of fluid flow, and the curvature decomposition leads to the generalisation of Raychaudhuri equation. In the latter case our framework generalises the kinematical notions of relativistic cosmology used in the context of general relativity [29] to the case of multi-dimensional fluids. In this way we have modelled the geometry of arbitrary brane worldsheet congruences in manifolds with metricity and torsion. In the context of our kinematical framework we have managed to obtain the generalisations of the Gauss–Codazzi and Raychaudhuri equations, as well as the law governing their generalised area change in a covariant form.

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